Convergence of a Kähler-Ricci flow

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Abstract

In this paper we prove that for a given Kähler-Ricci flow with uniformly bounded Ricci curvatures in an arbitrary dimension, for every sequence of times t_i converging to infinity, there exists a subsequence such that $(M, g(t_i + t)) \to (Y, \bar{g}(t))$ and the convergence is smooth outside a singular set (which is a set of codimension at least 4) to a solution of a flow. We also prove that in the case of complex dimension 2, without any curvature assumptions we can find a subsequence of times such that we have a convergence to a Kähler-Ricci soliton, away from finitely many isolated singularities.

1 Introduction

Let M be a compact Kähler manifold of dimension n with the Kähler metric $ds^2 = g_{i\bar{j}}dz^i d\bar{z}^j$. The Ricci curvature of this metric is given by the formula

$$R_{i\bar{j}} = \frac{-\partial^2}{\partial z^i \partial \bar{z}^j} \ln \det(g_{i\bar{j}}).$$

This implies that $\frac{\sqrt{-1}}{2\pi} R_{i\bar{j}} dz^i d\bar{z}^j$ is closed and its cohomology class is equal to the first Chern class $c_1(M)$ of M. We will assume that $c_1(M)$ is positive and that it is represented by a Kähler form. We will consider the complex version of Hamilton's Ricci flow equation of the following type

$$(g_{i\bar{j}})_t = g_{i\bar{j}} - R_{i\bar{j}} = \partial_i \bar{\partial}_j u, \tag{1}$$

where $g_{i\bar{j}}(t) = g_{i\bar{j}}(0) + \partial_i \bar{\partial}_j \phi$ and $\frac{d}{dt}\phi = u$. In [?] H.D. Cao proved that a solution of (1) exists for all times $t \in [0, \infty)$. A natural question that one can ask is what happens to a flow when time approaches infinity. Under which conditions will it converge? How can we describe the objects that we get in a limit? In this paper we will give partial answers to these questions.

In section 3 we will consider a Kähler-Ricci flow (1) with uniformly bounded Ricci curvatures. Our goal is to prove the following theorem.

Theorem 1. Assume we are given a flow (1). Assume that the Ricci curvatures are uniformly bounded, i.e. $|\mathrm{Ric}| \le C$ for all t. Then for every sequence $t_i \to \infty$ there exists a subsequence such that $(M, g(t_i + t)) \to (Y, \bar{g}(t))$ and the convergence is smooth outside a singular set S, which is at least of codimension four. Moreover, $\bar{g}(t)$ solves the Kähler-Ricci flow equation off the singular set.

In section 4 we will restrict ourselves to 2 dimensional complex orbifolds, without any curvature assumptions. We want to prove the following theorem.

Theorem 2. Let $g_{k\bar{j}}(t) = g_{k\bar{j}} - R_{k\bar{j}}$ be a Kähler-Ricci flow on a 2 dimensional complex, Kähler manifold. Then for every sequence $t_i \to \infty$ there exists a subsequence so that $(M, g(t_i + t)) \to (Y, \bar{g}(t))$ and $\bar{g}(t)$ is a Kähler-Ricci soliton.

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2 Background and notation

First of all, let us recall the definitions of Ricci solitons.

Definition 3. A solution $g_{i\bar{j}}$ to equation (1) on M is called Kähler-Ricci soliton if it moves along (1) under one-parameter family of automorphisms of M generated by some holomorphic vector field.

This means that

$$g_{i\bar{j}} - R_{i\bar{j}} = V_{i,\bar{j}} + V_{\bar{j},i},$$

for some holomorphic vector field $V = (V^i)$. In the case of limit solitons in Theorem 2, we will show that the vector fields come from the gradients of functions on M, i.e. that

$$g_{i\bar{j}} - R_{i\bar{j}} = f_{,i\bar{j}},$$

and $f_{,ij} = 0$ for some real valued function f on M. This condition is equivalent to a fact that $V = \nabla f$ is a holomorphic vector field.

Perelman's functional $\mathcal{W}$ for a flow (1) is

$$\mathcal{W}(g,f,\tau) = (4\pi\tau)^{-n}\int_M e^{-f}[2\tau(R+|\nabla f|^2)+f-2n]dV_g,$$

with a constraint that $(4\pi\tau)^{-n}\int_M e^{-f}dV_g = 1$. Perelman ([8]) has proved some very interesting properties of flow (1). We will list them in the following theorem.

Theorem 4 (Perelman). If (1) is a flow on a complex, Kähler, closed manifold M, then

1. $C^{1,\alpha}$ norms of functions $u(t)$ are uniformly bounded along the flow,
2. the scalar curvatures $R(t)$ and the diameters $\mathrm{diam}(M, g(t))$ are uniformly bounded along the flow,
3. a volume noncollapsing condition holds along the flow, i.e. there exists $C = C(g(0))$ such that $\mathrm{Vol}_t(B(p,r)) \ge Cr^n$.

We will need a theorem proved by Cheeger, Colding and Tian ([9]) in our further discussion and we will state it below for a reader's convenience.

Theorem 5 (Cheeger, Colding, Tian). If $\{M_i, g_i, p_i)\}$ converges to (Y, d, y) in pointed Gromov-Hausdorff topology, if $|\mathrm{Ric}|_{M_i} \le C$ and if $\mathrm{Vol}(B_1(p_i)) \ge C$ for all i, then the regular part $\mathcal{R}$ of Y is a $C^{1,\alpha}$-Riemannian manifold and at points of $\mathcal{R}$, the convergence is $C^{1,\alpha}$. Moreover the codimension of the set of singular points (which is a closed set in Y) is at least 4.

In the proof of Theorem 1 we will use Perelman's pseudolocality theorem ([7]).

Theorem 6 (Perelman). For every $\alpha > 0$ there exist $\delta > 0$, $\epsilon > 0$ with the following property. Suppose we have a smooth solution to the Ricci flow and assume that at $t = 0$ we have $R(x) \ge -r_0^{-2}$ and $\mathrm{Vol}(\partial\Omega)^n \ge (1-\delta)c_n\mathrm{Vol}(\Omega)^{n-1}$ for any $x, \Omega \subset B(x_0, r_0)$, where c_n is the euclidean isoperimetric constant. Then, $|\mathrm{Rm}|(x,t) \le \alpha t^{-1} + (\epsilon r_0)^{-2}$ whenever $0 < t \le (\epsilon r_0)^2$ and $\mathrm{dist}_t(x, x_0) < \epsilon r_0$.

Perelman proved this theorem for a case of unnormalized Ricci flow, but it can be easily modified for the case of a normalized Kähler-Ricci flow.

3 Kähler-Ricci flow with uniformly bounded Ricci curvatures

In this section we will consider a flow (1), with uniformly bounded Ricci curvatures. For any sequence $t_i \to \infty$, if $g_i(t) = g(t_i + t)$, the metrics $g_i(t)$ are uniformly equivalent to metrics $g_i(s)$ for s, t belonging to an interval of finite length. Moreover, the following proposition (in [4]) applies to metrics $g_i(t)$.

Proposition 7 (D. Glickenstein). Let $\{(M_i, g_i(t), p_i)\}_{i=1}^{\infty}$, where $t \in [0, T]$, be a sequence of pointed Riemannian manifolds of dimension n which is continuous in the t variable in the following way: for each $\delta > 0$ there exists $\eta > 0$ such that if $t_0, t_1 \in [0, T]$ satisfies $|t_0 - t_1| < \eta$ then

$$(1+\delta)^{-1} g_i(t_0) \le g_i(t_1) \le (1+\delta) g_i(t_0), \tag{2}$$

for all $i > 0$, and such that $\mathrm{Ric}(g_i(t)) \ge c g_i(t)$, where c does not depend on t or i. Then there is a subsequence $\{(M_i, g_i(t), p_i)\}_{i=1}^{\infty}$ and a 1-parameter family of complete pointed metric spaces $(X(t), d(t), x)$ such that for each $t \in [0, T]$ the subsequence converges to $(X(t), d(t), x)$ in the pointed Gromov-Hausdorff topology.

$(M_i, g_i(t))$ and $(M_i, g_i(0))$ are homeomorphic by Lipschitz homeomorphisms, and in [4] it has been showed be showed that $X(t)$ is homeomorphic to $X(0)$. If t_i is any sequence such that $t_i \to \infty$, Proposition 7 applies to $(M, g(t_i + t))$ for all i and all t belonging to a time interval of finite length.

For the moment we will restrict ourselves to the case of Kähler manifolds of complex dimension 2, and later we will show how it can be generalized to an arbitrary dimension. In the case of complex dimension 2, for every sequence $t_i \to \infty$ there is a subsequence $\{(M, g(t_i + t))\}$ converging to a compact orbifold $(Y, \bar{g}(t))$ with isolated singularities. This is due to the fact that L^2 norm of the curvature operator in the Kähler case can be uniformly bounded in terms of the first and the second Chern class of a manifold and its Kähler class. Combining Proposition 7 and Theorem 5 gives that $(Y, \bar{g}(t))$ is an 1 parameter family of orbifolds (it is even a Lipschitz family for t belonging to an interval of finite length), such that a regular part of $(Y, \bar{g}(t))$ is $C^{1,\alpha}$ manifold and the convergence $(M, g(t_i + t)) \to (Y, \bar{g}(t))$ takes place in $C^{1,\alpha}$ topology, away from the set of singular points (which is common for all orbifolds $(Y, \bar{g}(t))$). In the case of higher

dimensions, again by Proposition 7 and Theorem 5 we will have that $\{(M, g(t_i + t))\}$ converge to $(Y, \bar{g}(t))$ with a singular set $S \subset Y$ of codimension at least 4. $\mathcal{R} = Y \backslash S$ is an open $C^{1,\alpha}$ manifold and the convergence on $\mathcal{R}$ is in $C^{1,\alpha}$ norm. We will show later that the set S is common for $(Y, \bar{g}(t))$ for all t. The main tools in the proof of Theorem 1 will be Theorem 6 and Theorem A.1.5 of Cheeger and Colding that can be found in the appendix of [3].

We will now prove Theorem 1.

Proof. If the curvature does not blow up, we are done. Therefore, assume that the curvature does blow up. Let $t_i \to \infty$ be such that $Q_i = |\mathrm{Rm}|(p_i, t_i) \geq \max_{M\times[0,t_i]} |\mathrm{Rm}|(x, t)$ and $Q_i \to \infty$. We already know that since $|\mathrm{Ric}|(g(t)) \leq C$, there exists a subsequence $(M, g(t_i + t))$ converging to orbifolds $(Y, \bar{g}(t))$ in $C^{1,\alpha}$ norm off the set of singular points. Moreover, metrics $\bar{g}(t)$ are $C^{1,\alpha}$ off the singular set. We may assume that $\mathrm{Sing}(Y) = \{p\}$. Our goal is to show that we actually have C^∞ convergence off the singular point p, due to the fact that our metrics are changing with the Kähler-Ricci flow.

Adopt the notation of [9]. In general, a point $y \in Y$ is called regular, if for some k, every tangent cone at y is isometric to R^k. Denote a set of those points by $\mathcal{R}_k$ and let $\mathcal{R} = \cup_k \mathcal{R}_k$. Because of the noncollapsing condition that we have because of Theorem 4, we have that $\mathcal{R} = \mathcal{R}_n$. Let $\mathcal{R}_\epsilon = \{y \,|\, d_{GH}(B_1(y_\infty), B_1(0)) < \epsilon$ for every tangent cone $(Y_y, y_\infty)\}$, where $B_1(0)$ is a unit ball in R^n. Let $\mathcal{R}_{\epsilon,r}$ be a set of all points $y \in Y$ such that there exists x such that $(0, x) \in R^4 \times \{x\}$ and for some $u > r$ and every $s \in (0, u]$ $d_{GH}(B_s(y), B_s((0, x))) < \epsilon s$. $\mathcal{R}_\epsilon = \cup_r \mathcal{R}_{\epsilon,r}$.

Choose ϵ_P and δ_P as in Perelman's pseudolocality theorem. Choose $\epsilon' > 0$ such that $\delta_P > \epsilon'$ and $\epsilon' \leq \epsilon_0$, where ϵ_0 is such that $\mathcal{R} = \mathcal{R}_\epsilon$ for all $\epsilon \leq \epsilon_0$ (the existence of such an ϵ_0 is proved in section 7 of [3].

Fix a time $t = t_0$. Pick up any point $q \in Y \backslash \{p\}$. Then $q \in \cap_{\epsilon\leq\epsilon_0} \mathcal{R}_\epsilon$. Let $d = \mathrm{dist}_{\bar{g}(t_0)}(p, q)$.

Claim 8. There exist $\eta > 0$ and a sequence $q_i \in M$ such that $q_i \to q$, while $(M, g(t_i + t)) \to (Y, \bar{g}(t))$ as $i \to \infty$ for all t.

Proof. Assume that $q \in K \subset Y \backslash \{p\}$, where K is a compact set and let $r = \mathrm{dist}(K, p)$. For every t, $g(t_i + t)$ uniformly converge to $\bar{g}(t)$ on K. Let $\phi : K \to K_i$ be diffeomorphisms as in a definition of convergence of $(M, g(t_i + t))$ to $(Y, \bar{g}(t))$. Let $q_i(t_0) \in M$ be such that $\mathrm{dist}_{g(t_i+t_0)}(q_i(t_0), \phi_i(q)) < \epsilon$, for $i \geq i_0$. Since

the Ricci curvatures of $g(t)$ are uniformly bounded, there exists $\eta > 0$ so that $|t-s| < \eta$ implies that $|\mathrm{dist}_{g(t_i+s)}(x,y) - \mathrm{dist}_{g(t_i+t)}(x,y)| < \epsilon$, for all $x, y \in M$. Therefore,

$$\mathrm{dist}_{g(t_i+t)}(q_i(t_0), \phi_i(q)) \le \mathrm{dist}_{g(t_i+t_0)}(q_i(t_0), \phi_i(q)) + \epsilon < 2\epsilon,$$

for $i \ge i_0$ and for all $t \in [-\eta, \eta]$. Notice that η does not depend either on t_0 or q, but it depends on K, i.e. on its distance from p. Therefore, if we continue this process infinitely many times, considering $t_0 + \eta$ instead of t_0, etc. we get that the sequence $\{q_i\}$ will work for all times $t \ge 0$.

□

Lemma 9. For any regular point $q \in \mathcal{R}$ there exists i_0, η and $r > 0$ such that for all $B_{g(t_i+t)}(s, q') \subset B_{g(t_i+t)}(r, q_i(t))$ we have $\mathrm{Vol}_{g(t_i+t)} B_{g(t_i+t)}(s, q') \ge (1-\epsilon')s^n$, for all $i \ge i_0$ and all $t \in [t_0 - \eta, t_0 + \eta]$, where $q_i \in M$ is a sequence of points converging to q, while $(M, g(t_i + t)) \to (Y, \bar{g}(t))$.

Proof. For ϵ' find r and δ as in Theorem A.1.5 (i) and (ii) in [3]. For this δ (that now plays the role of ϵ in Theorem A.1.5 in [3]) find δ_1 and r_1 (by part (iii) of the same theorem), such that $x \in (\mathcal{WR})_{8\delta_1, r'}$ implies that

$$y \in \mathcal{R}_{\delta,s} \quad \forall y \in B_{r'}(x) \quad \forall s \le (1-\delta)r' - \mathrm{dist}_0(x,y), \quad r' \le r_1, \tag{3}$$

where a distance is measured in metric $\bar{g}(t_0)$. We may assume that $r_1 < d$, because otherwise we can decrease r_1. Take any sequence $\delta_i \to 0$ as $i \to \infty$. We can choose a sequence r_i such that $q \in \mathcal{R}_{\delta_i, r_i}$, since $q \in \mathcal{R}_{\delta_i}$. We claim that $q \in \mathcal{R}_{\delta_1, r}$, for some $r < r_1$. In order to prove that, we may assume $r_i \to 0$ (otherwise if $r_i \ge \kappa$ for all i, $d_{GH}(B_l(q), B_l(0)) \le l\delta_i \to 0$ for all $l \le \kappa$ and therefore we would have $\mathrm{Vol}B_l(q) = \mathrm{Vol}B_l(0)$ for all $l \le \kappa$, and $q \in \mathcal{R}_{\delta_1, s}$ for some $s < r_1$, by Theorem A.1.5, part (i) in [3]. Therefore, there exist $\delta'' < \delta_1$ and $r'' < r_1$ such that $q \in \mathcal{R}_{\delta'', r''}$. This implies $q \in \mathcal{R}_{\delta_1, r''}$, since $\delta'' < \delta_1$. This is true in metric $\bar{g}(t_0)$.

Claim 10. There exist $\eta > 0$ and i_0 such that $q_i \in \mathcal{R}_{\delta_1, r''}$ for all metrics $g(t_i + t)$ for $i \ge i_0$ and $t \in [t_0 - \eta, t_+ \eta]$.

Proof. $q \in \mathcal{R}_{\delta_1, r''}$ and therefore,

$$d_{GH}(B_s(x, 0), B_s(q, t_0)) < s\delta_1, \tag{4}$$

for some $s < r''$. We can substitute space $\{x\} \times \mathrm{R}^4$ by R^4 only and therefore we can write just $B_s(0)$ instead of $B_s(x,0)$. Since the Ricci curvatures of $g(t)$ are uniformly bounded, there exists η such that $|t - t_0| < \eta$ implies that

$$d_{GH}(B_{g(t_i+t)}(q_i, s), B_{g(t_i+t_0)}(q_i, s)) < \delta_1 s. \tag{5}$$

Since $g(t_i + t_0)$ converges to $\bar{g}(t_0)$ uniformly, away from a singular point p, there exists i_0 (depending on $\delta_1 s$ and a compact set K) such that for $i \ge i_0$

$$d_{GH}(B_{g(t_i+t_0)}(q_i, s), B_{\bar{g}(t_0)}(q, s)) < \delta_1 s. \tag{6}$$

Combining estimates (4), (5) and (6), together with an approximate triangle inequality for Gromov-Hausdorff distance we get

$$d_{GH}(B_{g(t_i+t)}(q_i, s), B_s(0)) < 4\delta_1 s,$$

for all $i \ge i_0$ and all $t \in [t_0 - \eta, t_0 + \eta]$. This implies that $q_i \in \mathcal{WR}_{8\delta_1, r''}$, for all $i \ge i_0$ and all $t \in [t_0 - \eta, t_0 + \eta]$.

$\square$

Combining Claim 10 and part (iii) of Theorem $A.1.5$ in [3], we get that $q' \in \mathcal{R}_{\delta,s}$, for all $q' \in B_{g(t_i+t)}(q_i, r'')$, $s \le (1-\delta)r'' - \mathrm{dist}_{t_i+t}(q_i, q')$, for all $i \ge i_0$ and $t \in [t_0 - \eta, t_0 + \eta]$. Part (ii) of Theorem $A.1.5$ in [3] gives that

$$\mathrm{Vol} B_{g(t_i+t)}(s, q') \ge (1 - \epsilon')\mathrm{Vol} B_s(0), \tag{7}$$

for all $q' \in B_{g(t_i+t)}(r'', q)$ and $s \le (1-\delta)r'' - \mathrm{dist}_{g(t_i+t)}(q_i, q')$. By reducing r'' we get that there exists r'' such that the estimate (7) holds for all $q' \in B_{g(t_i+t)}(r'', q)$ and all s such that $B_{g(t_i+t)}(s, q') \subset B_{g(t_i+t)}(r'', q)$, for $i \ge i_0$ and $t \in [t_0 - \eta, t_0 + \eta]$.

$\square$

Choose r, i_0 and η as in the claim above (for our regular point q that we have fixed earlier). Reduce r'' if necessary, so that $(\epsilon' r'')^2 < \eta$. Since $1 - \epsilon' > 1 - \delta_P$, and since for every ball $B_{g(t_i - (\epsilon' r'')^2/2)}(q', s) \subset B_{g(t_i - (\epsilon' r'')^2/2)}(q_i, r'')$, we have that $\mathrm{Vol}_{g(t_i - (\epsilon' r'')^2/2)} B_s(q') \ge (1 - \delta_P) s^n c_n$, by Perelman's pseudolocality Theorem 6

$$|\mathrm{Rm}|(x,t) \le \frac{1}{(\epsilon' r'')^2} + (\epsilon' r'')^2,$$

for all $x \in B_{g(t)}(q_i, \epsilon' r'')$ and for every $t \in [t_i - (\epsilon' r'')^2/2, t_i + (\epsilon' r'')^2/2]$. We have that $g_i(t) = g(t_i + t)$ is a sequence of Ricci flows with uniformly bounded curvatures for $t \in [-(\epsilon' r'')^2/2, (\epsilon' r'')^2/2]$ on balls $B_{g_i(t)}(q_i, \epsilon' r'')$. This together with the volume noncollapsing condition and Hamilton's compactness theorem give that the convergence of the sequence of our metrics is smooth, and $\bar{g}(t)$ are smooth metrics on $B_{\bar{g}(t)}(q, \epsilon' r'')$, for $t \in [0, (\epsilon' r'')^2/2]$. Repeating the procedure described above infinitely many times, to time intervals translated by $(\epsilon' r'')^2/2$ (considering $t_0 + (\epsilon' r'')^2/2$ instead of t_0, etc.) and applying diagonalization method to a sequence of times t_i (since for every step of length $(\epsilon' r'')^2/2$ we have to extract a subsequence of a subsequence), we get that $\bar{g}(t)$ are smooth metrics on $B(q, \epsilon' r'')$ for all times $t \geq 0$ (we can take $t_0 = 0$) and that $g(t_i + t) \to \bar{g}(t)$ smoothly on $B_{\bar{g}(t)}(q, \frac{\epsilon' r''}{2})$ for all times $t \geq 0$. We will use the fact that the Ricci tensor is uniformly bounded to show that we can extend the previous result from a ball to any compact set $K \subset Y$. By a definition of convergence, that will mean $(M, g(t_i + t)) \to (Y, \bar{g}(t))$ smoothly, away from the set of singular points.

Take a compact set $K \subset Y \backslash S$, where S is a set of singular points on $(Y, \bar{g}(t))$. It is the same set for all singular metrics $\bar{g}(t)$. Let $\phi_i : K \to K_i$ be a sequence of diffeomorphisms from a definition of convergence of metrics $g_i(0)$ to a metric $\bar{g}(0)$.

$|\mathrm{Ric}|(t) \leq C$ for all t by the assumtion of the main theorem. We have proved that $\bar{g}(t)$ is 1-parameter family of metrics on Y. Moreover, $\bar{g}(t)$ satisfies the Kähler-Ricci flow equation away from the singular points.

Claim 11. There exist $\delta > 0$, a subsequence t_i and $C_1 = C_1(K)$ such that $|\mathrm{Rm}|(g(t_i + t)) \leq C$ on K_i for all $t \in [t_0, t_0\delta]$ and all t_0.

Proof. Fix t_0. For every $q \in Y$ we can choose $r_q > 0$, η_q and i_q as in Lemma 9. Look at the collection of balls $B_{\bar{g}(t_0)}(q, (\epsilon' r_q)/4)$ covering K. Since K is compact we can consider only finitely many of them covering K. Denote their centres by $q_1, q_2, \dots q_N$. Since $\bar{g}(t)$ solves the equation (1) and since the Ricci curvatures of $\bar{g}(t)$ are uniformly bounded on $Y \backslash \{p\}$, there exists $A > 0$ so that the balls $B_{\bar{g}(t)}(q_i, (\epsilon' r_{q_i})/2)$ cover K, for $t \in [t_0 - A, t_0 + A]$. Let $r_1 = \min\{r_{q_1}, r_{q_2}, \dots, r_{q_N}\}$, $\eta_1 = \min\{\eta_{q_1}, \dots, \eta_{q_N}\}$ and $i_1 = \max\{i_{q_1}, \dots i_{q_N}\}$. Then $|\mathrm{Rm}|(x, t + t_i) \leq \frac{1}{(\epsilon' r_1)^2} + (\epsilon' r_1)^2 = C_1(K)$ for all $x \in B_{g(t_i+t)}(q_i^j, \epsilon' r_{q_j})$, all $i \geq i_1$ and all $t \in [0, \min\{\eta_1, (\epsilon' r_1)^2/2\}]$, where q_i^j are the sequences of points such that $B_{g_(t_i+t)}(q_i^j, r_{q_j}) \to B_{\bar{g}(t)}(q_j, r_{q_j})$ while $(M, g(t_i + t)) \to (Y, \bar{g}(t))$. Let

$\delta = \min\{\eta_1, (\epsilon' r_1)^2/2, A\}]$.

The balls $B_{\bar{g}(t)}(q_j, (\epsilon' r_{q_j})/2)$ for $1 \le j \le N$ cover K. A definition of convergence gives that there exists $i_0 \ge i_1$ so that $B_{g(t_i)}(q_i^j, \frac{2\epsilon' r_{q_j}}{3})$ cover K_i for all $i \ge i_0$. We can assume that δ is small enough so that

$$B_{g(t_i)}(q_i^j, (r_{q_j} - a)\epsilon') \subset B_{g(t_i+s)}(q_i^j, r_{q_j}\epsilon'/2),$$

for $a < \frac{r_1}{3}$ so that $r_{q_j} - a > r_{q_j} - \frac{r_1}{3} > \frac{2r_{q_j}}{3}$ and therefore $B_{g(t_i)}(q_i^j, (\epsilon'(2r_{q_j})/3) \subset B_{g(t_i)}(q_i^j, (r_{q_i^j} - a)\epsilon')$. Since the balls $B_{g(t_i)}(q_i^j, (2r_{q_j}\epsilon')/3)$ cover K_i, so do balls $B_{g(t_i+s)}(q_i^j, r_{q_j})$ for all $i \ge i_0$ and all $s \in [t_0-\delta, t_0+\delta]$. Therefore, $|\mathrm{Rm}|(x, t_i+s) \le C_1$ for all $i \ge i_1$ and all $x \in K_i$ and all $s \in [t_0 - \delta, t_0 + \delta]$. Therefore we actually can extract a subsequence t_i such that the pullbacks of metrics $g(t_i + s)$ converge to a solution of the Kähler-Ricci flow uniformly on $K \times [t_0 - \delta, t_0 + \delta]$. □

Applying the method from the previous claim to a sequence $t_i + \delta$ instead of a sequence t_i we can find a subsequence such that $g(t_i + s) \to \bar{g}(t)$ smoothly and uniformly on $K \times [t_0, t_0 + 2\delta]$, since our choice of δ does not depend on an initial time, but on a chosen compact set $K \subset Y$ and a uniform bound on the Ricci tensor. Repeating this infinitely many times and diagonalizing the sequence t_i, we get a sequence t_i such that $g(t_i + s) \to \bar{g}(t)$ smoothly converge on all compact subsets of $K \times [0, \infty)$. We can choose a countable sequence of compact sets L_k exhausting $Y \backslash S$. We can find a subsequence of t_i for each L_k, so that the above that we have proved for any compact set $K \subset Y \backslash S$, applies to L_k as well. By a diagonalization procedure applied to t_i we can get a subsequence so that $(M, g(t_i + t) \to (Y, \bar{g}(t))$ for all $t \ge 0$, where $\bar{g}(t)$ is a solution to the Kähler-Ricci flow away from the set of singular points. The convergence is in the sense that for every compact set $K \subset Y \backslash S$ there exist diffeomorphisms $\phi_i : K \to K_i$, where $K_i \subset M$ are compact and $\phi_i^* g(t_i + t) \to \bar{g}(t)$, uniformly and smoothly on all compact subsets of $K \times [0, \infty)$.

□

In the proof of Theorem 1 we assumed that the complex dimension was 2. The proof above generalizes to an arbitrary dimension easily. Since the Ricci tensor is uniformly bounded along the flow, for every sequence $t_i \to \infty$ there exists a subsequence so that $(M, g(t_i + t)) \to (Y(t), \bar{g}(t))$ and the convergence is smooth

outside a set $S(t)$ of codimension at least 4. As above, it easily follows that $Y = Y(t)$ for all t. We should only check that $S(t) = S(s)$ for any $s, t \in [0, \infty)$.

Lemma 12. $S(s) = S(t)$ for any $s, t \in [0, \infty)$.

Proof. It is enough to prove: $\exists a > 0$ such that for $|s - t| < a$ $S(s) = S(t)$.

Choose $\epsilon > 0$ such that $\mathcal{R}_\epsilon(s) = \mathcal{R}(s)$ and $\mathcal{R}_\epsilon(t) = \mathcal{R}(t)$, for $|s - t| < a$, where we will choose a later. Assume there exists $q \in S(t)\backslash S(s)$. That implies $q \in \mathcal{R}(s)$. For $\epsilon > 0$ choose $\epsilon' = \epsilon'(\epsilon, n) > 0$ and $r' = r'(\epsilon, n)$ so that Theorem $A.1.5$ in [3] holds. Then the following claim holds for q.

Claim 13. There exist i_0 and $r < r'$ such that for all $B_{g(t_i+s)}(q', u) \subset B_{g(t_i+s)}(q_i, r)$ we have $\mathrm{Vol}_{g(t_i+s)} B_{g(t_i+s)}(q', u) \geq c_n(1 - \epsilon'/2)u^n$, for all $i \geq i_0$, where $q_i \in M$ is a sequence of points converging to q, while $(M, g(t_i + s)) \to (Y, \bar{g}(s))$.

The proof of this claim is the same as the proof of Lemma 9.

Since the Ricci tensors are uniformly bounded along the flow, we have a good control on the volumes and the sizes of balls in metrics at different times, when the considered time interval is sufficiently small. Similarly as in [5] we can find sufficiently small $a > 0$ such that $|s - t| < a$, for any $u < r$ implies that

$$\mathrm{Vol}_{g(t_i+t)} B_{g(t_i+t)}(q, u) \geq \sqrt{(1 - \frac{\epsilon'}{2})} \mathrm{Vol}_{g(t_i+s)} B_{g(t_i+s)}(q, u) \geq (1 - \frac{\epsilon'}{2}),$$

$$B_{g(t_i+s)}(q, u\tilde{r}) \subset B_{g(t_i+t)}(q, u),$$

where $\tilde{r} = \frac{1}{1 + (e^{2C|s-t|} - 1)^{\frac{1}{2}})}$ and we can choose a small enough, so that $\tilde{r}^n > \sqrt{1 - \frac{\epsilon'}{2}}$. Finally, since $\mathrm{Vol}_{g(t_i+s)} B_{g(t_i+s)}(q, u) \geq c_n(1 - \frac{\epsilon'}{2})u^n$, we get that

$$\mathrm{Vol}_{g(t_i+t)} B_{g(t_i+t)}(q, u) \geq (1 - \epsilon'/2)^2 u^n c_n \geq (1 - \epsilon') c_n u^n,$$

i.e. $q \in \mathcal{R}_{\epsilon, r/2} \subset \mathcal{R}_\epsilon = \mathcal{R}(t)$. This means that q can not be in $S(t)$ and we get a contradiction. We can repeat the procedure above infinitely many times to get that $S(t) = S(s)$ for all $s, t \in [t_0, t_0 + a]$ and all $t_0 \geq 0$, i.e. $S(t) = S$ for all $t \geq 0$.

Having Lemma 12 we can repeat the proof of Theorem 1 for complex dimension 2, to get that theorem is actually true for all dimensions.

$\square$

4 Kähler-Ricci soliton as a limit

Assume that $(Y, \bar{g})$ is a complex n-dimensional orbifold with finitely many singularities. Assume without loss of generality that p is its only singular point. Analogously to the case of compact manifolds we can show that $\mu(\bar{g}(t), \frac{1}{2}) = \inf_{\{f|(2\pi)^{-n}\int_M e^{-f}=1\}}$ is achieved and that

$$2\Delta f - |\nabla f|^2 + R + f - 2n = \mu(\bar{g}(t), \frac{1}{2}),$$

on $Y\backslash\{p\}$ where $f(t)$ is a function such that $\mu(\bar{g}(t), \frac{1}{2}) = W(\bar{g}(t), f(t), \frac{1}{2})$. Before we start proving Theorem 2, we will fisrt prove the following proposition.

Proposition 14. Let (M, g_i) be a sequence of smooth, closed manifolds, with uniformly bounded scalar curvatures, converging to an orbifold $(Y, \bar{g})$ with a singular point p. Assume that $|\mu(g_i, \frac{1}{2})| \le C$ for all i. Then $\lim_{i\to\infty} \mu(g_i, \frac{1}{2}) = \mu(\bar{g}, \frac{1}{2})$.

Proof. Fix $\epsilon > 0$. Similarly as in the smooth case we can show that $|f_i|_{C^{2,\alpha}} \le C$ on $Y\backslash\{p\}$ (by weak regularity theory applied to $\tilde{f}_i = e^{-\frac{f_i}{2}}$ we get that $W^{3,p}$ norms of $\tilde{f}_i$ are uniformly bounded and Sobolev embedding theorems applied to $\tilde{f}_i$ and $\Omega_k = Y\backslash B(p, \frac{1}{k})$ give uniform $C^{2,\alpha}$ bounds, i.e. $|\tilde{f}_i|_{C^{2,\alpha}(\Omega_k)} \le C$, where C does not depend on k).

$\mu(\bar{g}, \frac{1}{2}) = \int_Y F d_{\bar{g}}$, where $F = (2\pi)^{-n} e^{-f}(|\nabla f|^2 + R + f - 2n)$. F is an integrable function and therefore there exists some $r > 0$ such that $\int_{B(p,2r)} |F| dV_{\bar{g}} < \epsilon$. Then:

$$\begin{aligned}\mu(\bar{g}, \frac{1}{2}) &> \int_{Y\backslash B(p,2r)} F dV_{\bar{g}} - \epsilon \\ &= \int_{Y\backslash B(p,2r)} F(dV_{\bar{g}} - dV_{\tilde{g}_i}) + \int_{Y\backslash B(p,2r)} F dV_{\tilde{g}_i} - \epsilon,\end{aligned}$$

for $\tilde{g}_i = \phi_i^* g_i$ ($\phi_i : Y\backslash B(p, 2r) \to M\backslash B(p_i, r)$ are diffeomorphisms from a definition of convergence). Since $|F| \le C$ on $Y\backslash\{p\}$ and since $\tilde{g}_i$ converge uniformly to $\bar{g}$ on $Y\backslash B(p, 2r)$, there exists i_0 such that for all $i \ge i_0$

$$
\begin{aligned}
\mu(\bar{g}, \frac{1}{2}) &> \int_{Y \backslash B(p,2r)} F dV_{\tilde{g}_i} - 2\epsilon \\
&= \int_{\phi_i(Y\backslash B(p,2r))} \phi^* F dV_{g_i} - 2\epsilon \\
&= \int_{M \backslash U_r^i} \phi_i^* F dV_{g_i} - 2\epsilon,
\end{aligned}
$$

where $p_i \in U_r^i$ and $B(p_i, r) \subset U_r^i \subset B(p_i, 2r)$ (we can always choose big i_0 such that these inclusions hold for $i \geq i_0$). Let $\tilde{f}_i = \phi_i^* f$. $\int_Y e^{-f} dV_{\bar{g}} = (2\pi)^n$ and therefore $(2\pi)^n - \epsilon < \int_{Y\backslash B(p,2r)} e^{-f} dV_{\bar{g}} < (2\pi)^n + \epsilon$ (we can decrease $r > 0$ so that this holds, since f is bounded on $Y \backslash \{p\}$). For the same reasons as above, for $i \geq i_0$ we have that $(2\pi)^n - 2\epsilon < \int_{Y\backslash B(p,2r)} e^{-f} dV_{\tilde{g}_i} < (2\pi)^n + \epsilon$. Therefore, for $i \geq i_0$ (we increase i_0 if necessary)

$$
(2\pi)^n - \epsilon < \int_{\phi_i(Y\backslash B(p,2r))} e^{-\tilde{f}_i} dV_{g_i} < (2\pi)^n + 2\epsilon.
$$

Choose $\eta_i = 1$, outside $B_{g_i}(p_i, 3r)$ and 0 inside the ball $B_{g_i}(p_i, 2r)$ so that $|\nabla_i \eta_i| \leq \frac{1}{r}$. Let $\bar{f}_i = \eta_i \tilde{f}_i$. For r small enough and i_0 big enough we have

$$
|\int_{\phi_i(Y\backslash B(p,2r))} e^{-\tilde{f}_i} dV_{g_i} - \int_{\phi_i(Y\backslash B(p,2r))} e^{-\bar{f}_i} dV_{g_i}| < \epsilon,
$$

which implies $(2\pi)^n - C\epsilon < \int_M e^{-\bar{f}_i} dV_{g_i} < (2\pi)^n + C\epsilon$. Modify each $\bar{f}_i$ by a small constant so that $\int_M e^{-\bar{f}_i} V_{g_i} = 1$. Furthermore,

$$
\begin{aligned}
& |\int_M |\nabla \bar{f}_i|^2 e^{-\bar{f}_i} dV_{g_i} - \int_M |\nabla \tilde{f}_i|^2 e^{-\tilde{f}_i} dV_{g_i}| \leq \\
\leq \;& C \int_{B(p_i,3r)\backslash B(p_i,2r)} (|\nabla \eta_i|^2 \tilde{f}_i^2 + \eta_i^2 |\nabla \tilde{f}_i|^2 + |\nabla \tilde{f}_i|^2) dV_{g_i} \\
\leq \;& C(\int_{B(p_i,3r)\backslash B(p_i,2r)} \frac{1}{r^2} dV_{g_i} + \int_{B(p_i,3r)\backslash B(p_i,2r)} |\nabla \tilde{f}_i|^2 dV_{g_i}) < \epsilon,
\end{aligned}
$$

for small enough r, since there exists constant C so that $\mathrm{Vol}_{g(t)} B(x,r) \leq Cr^4$ for all $r \leq r_0$ and all t. This simple follows from the fact that $\frac{\mathrm{Vol} B(p,r)}{V_{-C}(r)} \leq \frac{\mathrm{Vol} B(p,\delta)}{V_{-C}(\delta)}$, where $V_{-C}(r)$ is a volume of a ball of radius r in a simply connected space of constant curvature $-C$, the fact that $\lim_{\delta\to 0} \frac{\mathrm{Vol} B(p,\delta)}{V_{-C}(\delta)} = w_n$ and $\lim_{r\to 0} \frac{V_{-C}(r)}{r^4} = w_n$ (w_n is a volume of a euclidean unit ball). Similarly we can estimate other terms that appear in $\phi^* F$ and finally we can get

$$\mu(\bar{g}, \frac{1}{2}) > (2\pi)^{-n} \int_M e^{-\bar{f}_i}[|\nabla \bar{f}_i|^2 + R + \bar{f}_i - 2n]dV_{g_i} - k\epsilon,$$

for big enough i. Therefore, $\mu(\bar{g}, \frac{1}{2}) > \mu(g_i, \frac{1}{2}) - k\epsilon$ for $i \geq i_0$, for some constant k. In a very similar manner as above we can get that $\mu(g_i, \frac{1}{2}) > \mu(g_i, \frac{1}{2}) - k\epsilon$ for $i \geq i_0$, since f_i satisfy

$$2\Delta f_i - |\nabla f_i|^2 + R(g_i) + f_i - 2n = \mu(g_i, \frac{1}{2}),$$

where $C_1 \leq \mu(g_i, \frac{1}{2}) \leq C_2$.

□

Remark 15. The previous proposition holds for a Riemannian manifold of an arbitrary dimension.

From now on restrict ourselves to a case of a Kähler-Ricci flow in complex dimension 2. Tian showed in [10] that in complex dimension 2 we do not need any curvature assumptions to show that if $g(t)$ is a Kähler-Ricci flow then for every sequence $t_i \to \infty$ there exists a subsequence such that $(M, g(t_i + t)) \to (Y, \bar{g}(t))$, in the orbifold sense, where $(Y, \bar{g}(t))$ are the orbifolds with only finitely many singularities. Our goal is to prove Theorem 2, i.e.

Theorem 16. Let $(g_{k\bar{j}})_t = g_{j\bar{k}} - R_{j\bar{k}} = \partial_j\bar{\partial}_k u$ be a Kähler-Ricci flow on a 2-dimensional, complex, Kähler manifold. Then for every sequence $t_i \to \infty$, there exists a subsequence so that $(M, g(t_i + t)) \to (Y, \bar{g}(t))$ and $\bar{g}(t)$ is a Kähler-Ricci soliton, i.e.

$$g_{i\bar{j}} - R_{i\bar{j}} = \partial_i\partial_{\bar{j}}\bar{u},$$

where $u_{ij} = u_{\bar{i}\bar{j}} = 0$ and $\bar{u}(t)$ is a minimizer for $\mathcal{W}$ with respect to $\bar{g}(t)$.

Proof. Fix $r > 0$. Then the pullbacks of metrics $g(t_i + t)$ converge uniformly to $\bar{g}(t)$ on $Y \backslash B_{\bar{g}(t)}(p, r/3) \times [0, A]$, for any finite A. Therefore, there exists i_0 so that $|\mathrm{Ric}|(g(t_i + t)) \leq C = C(r)$ and $|\mathrm{Rm}|(g(t_i + t)) \leq C$ for all $i \geq i_0$ on $\phi_i(Y \backslash B_{\bar{g}(t)}(p, r/3))$, where ϕ_i are diffeomorphisms from a definition of convergence and for all $t \in [0, A]$.

(*)First of all we will make an appropriate choice of cut off functions. Choose small $\delta < A$ so that

$$B_{g(t_i+s)}(p_i, r/2) \subset B_{g(t_i)}(p_i, r),$$

and

$$B_{g(t_i)}(p_i, 2r) \subset B_{g(t_i+s)}(p_i, 5r/2),$$

for all $s \in [0, \delta]$, all $r > 0$ and all $i \geq i_0$. This is possible, since the uniform bounds on the Ricci curvatures give a good control over the distances, the sizes of balls and the norms of vectors at different times. This control does not depend on either the choice of a point or time.

Let ξ_i be a sequence of cut off functions such that $\xi_i = 1$ outside the ball $B_{g(t_i)}(p_i, 2r)$, $\xi_i = 0$ in $B_{g(t_i)}(p_i, r)$ and the norms of the derivatives of ξ in metrics $g(t_i)$ are uniformly bounded in i by a constant that dependes on r. We can decrease δ if necessary, so that for $i \geq i_0$, norms of the derivatives of ξ_i in metrics $g(t_i + s)$ are uniformly bounded in i and $s \in [0, \delta]$. Then we have that our cut off functions xi_i acutually satisfy

$$\xi_i = \begin{cases} 0 & \text{if } x \in B_{g(t_i+s)}(p_i, r/2) \\ 1 & \text{outside } B_{g(t_i+s)}(p_i, 5r/2) \end{cases}$$

for all $s \in [0, \delta]$ and all $i \geq i_0$.

For every t, choose a minimizer for Perelman's functional $\mathcal{W}$ with respect to $g(t)$. Denote it by f_t. Flow it backwards. Let $u_t = e^{-f_t}$. Let $t'_i = t_i + \delta$. Then

$$\frac{d}{ds} u_{t'_i}(s) = -\Delta u_{t'_i}(s) + (n - R) u_{t'_i}(s).$$

Following Perelman's computation we get

$$\frac{d}{ds} \mathcal{W}(g(t_i + s), f_{t'_i}(s), 1/2) = (2\pi)^{-n} \int_M |R_{k\bar{j}} + \nabla_k \bar{\nabla}_j f_{t'_i} - g_{k\bar{j}}|^2 (g(t_i + s) dV_{g(t_i+s)}.$$

If we integrate it over $s \in [0, \delta]$, we get

$$\begin{aligned} \mu(g(t_i + \delta), 1/2) - \mu(g(t_i), 1/2) &\geq \mathcal{W}(g(t'_i), f_{t'_i}, 1/2) - \mathcal{W}(g(t_i), f_{t'_i}(t_i), 1/2) \\ &= (2\pi)^{-n} \int_0^\delta \int_M |\nabla \bar{\nabla} (f_{t'_i} - u)|^2 (t_i + s) dV_{g(t_i+s)}. \end{aligned}$$

$\mu(g(t), \frac{1}{2}) \leq \mathcal{W}(g(t), u(t), 1/2) = (2\pi)^{-n} \int_M e^{-u} (|\nabla u|^2 + R + u - 2n) dV_{g(t)} \leq C$, because of Theorem 4. Combining this with Perelman's montonicity formula give that there exists a finite $\lim_{t \to \infty} \mu(g(t), 1/2)$ and therefore

$$\lim_{i\to\infty} |\nabla\bar{\nabla}(f_{t_i'} - u)|(t_i + s) = 0,$$
for almost all $x \in M$ and almost all $s \in [0, \delta]$.

Moreover,

$$\frac{d}{ds}\xi_i u_{t_i'}(s) = -\xi_i \Delta u_{t_i'}(s) + \xi_i(n - R)u_{t_i'}(s),$$
for $s \in [0, \delta]$. As in [6] we can get that

$$|u_{t_i'}(t_i + s)|_{C^{2,\alpha}(M\backslash B_{g(t_i+s)}(p_i,5r/2))} \leq C(r),$$

where $C(r)$ is a constant that depends on $r > 0$ which we have fixed at the beginning. Since the curvature is uniformly bounded on $M\backslash B_{g(t_i+s)}(p_i, 5r/2)$ for all $i \geq i_0$ and all $s \in [0, \delta]$, we have also that

$$|u(t_i + s)|_{C^{2,\alpha}(M\backslash B_{g(t_i+s)}(p_i,5r/2))} \leq C(r).$$

We can now extract a subsequence t_i such that $u_{t_i'}(t_i + s) \stackrel{C^{2,\alpha}}{\to} \bar{u}_1(s)$ and $u(t_i + s) \stackrel{C^{2,\alpha}}{\to} \bar{u}(s)$, on $Y\backslash B_{\bar{g}(s)}(p, 5r/2)$, uniformly in $s \in [0, \delta]$. Because of those $C^{2,\alpha}$ estimates, we have that

$$\nabla\bar{\nabla}(\bar{u} - \bar{f}) = 0,$$
i.e. $\Delta\bar{f} = \Delta\bar{u}$ on $Y\backslash B_{\bar{g}(s)}(p, 5r/2)$, where $\bar{u} = e^{-\bar{f}}$.

We can consider instead of $r > 0$ a sequence $r_k = 1/k \to 0$. Choosing a subsequence of a subsequence of a sequence t_i for each k-th step, using the uniqueness of a limit and a diagonalization method we can extract a subsequence t_i such that $u_{t_i'}(t_i + s) \stackrel{C^{2,\alpha}}{\to} \bar{f}(s)$ and $u(t_i + s) \stackrel{C^{2,\alpha}}{\to} \bar{u}(s)$ uniformly on compact subsets of $Y\backslash\{p\} \times [0, \delta]$. We also know that $\Delta\bar{f}(s) = \Delta\bar{u}(s)$ on $Y\backslash\{p\}$ and therefore $\bar{f}(s) = \bar{u}(s)$, since both of them satisfy the same integral normalization condition

$$\int_Y e^{-\bar{f}} dV_s = \int_Y e^{-\bar{u}} dV_s = (2\pi)^n,$$
and since Y is a compact orbifold. Because of the uniform convergence on compact sets we have that $\bar{u}(s) = \bar{f}(s)$ satisfy

$$\frac{d}{ds}\bar{f}(s) = -\Delta\bar{f} + |\nabla\bar{f}|^2 - R + n.$$

On the other hand $\bar{u}(s)$ satisfy

$$\frac{d}{ds}\bar{u} = \Delta\bar{u} + \bar{u} + a,$$
$$\Delta\bar{u} = n - R.$$

Therefore,

$$\frac{d}{ds}\bar{u}(s) = |\nabla\bar{u}|^2(s),$$

i.e.

$$\Delta\bar{u} - |\nabla\bar{u}|^2 + \bar{u} = -a.$$

$f_{t_i'}(t_i') = f_{t_i'}$ is a minimizer for $\mathcal{W}$ with respect to $g(t_i')$ and it converges to a minimizer for $\mathcal{W}$ with respect to a limit metric $\bar{g}(\delta)$, which is a consequence of Proposition 14 (the arguments for this are similar as in [6]). Therefore, $\bar{u}(\delta)$ is a minimizer for $\mathcal{W}$ with respect to $\bar{g}(t)$.

Claim 17. $\bar{a}(s) = a$, for all $s \in [0,\delta]$, where $\bar{a}(s) = -(2\pi)^{-n}\int_Y \bar{u}e^{-\bar{u}}dV_s$.

Proof. We can repeat everything that we have done before, replacing δ with any $t \in (0,\delta]$ to conclude that $\bar{u}(t)$ is a minimizer for $\mathcal{W}$ with respect to $\bar{g}(t)$ for $t \in (0,\delta]$.

$$\mu(\bar{g}(t),1/2) = (2\pi)^{-n}\int_Y (|\nabla\bar{u}|^2 - \Delta\bar{u} + \bar{u} - n)e^{-\bar{u}}dV_{\bar{g}(s)} = -\bar{a}(s) - n.$$

On the other hand there exists a finite $\lim_{t\to\infty}\mu(g(t),1/2)$ and therefore $\mu(\bar{g}(t),1/2) = \mu(\bar{g}(s),1/2)$ for all $s,t \in (0,\delta]$, i.e. $\bar{a}(t) = a$ for some constant $\bar{a}$ for all $s \in (0,\delta]$. Because of the continuity, $\bar{a}(t) = a$ for all $t \in [0,\delta]$.

□

Claim 18. $\bar{u}_{ij} = \bar{u}_{\bar{i}\bar{j}} = 0$ for all $t \in [0,\delta]$.

Proof.

$$\Delta\bar{u} - |\nabla\bar{u}|^2 + \bar{u} = -a.$$

From here we see $(\Delta\bar{u})^2 = \Delta\bar{u}|\nabla\bar{u}|^2 - \bar{u}\Delta\bar{u} - a\Delta\bar{u}$, i.e.

$$\int_M \Delta\bar{u}(|\nabla\bar{u}|^2 - \Delta\bar{u})dV_s = \int_M \bar{u}\Delta\bar{u} = -\int_M |\nabla\bar{u}|^2 dV_s \tag{8}$$

We have the following evolution equations:

$$\frac{d}{dt}|\nabla\bar{u}|^2 = \Delta|\nabla\bar{u}|^2 - |\nabla\nabla\bar{u}|^2 - |\nabla\bar{\nabla}\bar{u}|^2 + |\nabla\bar{u}|^2.$$

$$\frac{d}{dt}\Delta\bar{u} = \Delta^2\bar{u} + \Delta\bar{u} - |\nabla\bar{\nabla}\bar{u}|^2.$$

If we subtract the second equation from the first one and if we integrate what we get along M, we get

$$\int_M \frac{d}{dt}(|\nabla\bar{u}|^2 - \Delta\bar{u}) = -\int_M |\nabla\nabla\bar{u}|^2 + \int_M |\nabla\bar{u}|^2,$$

i.e.

$$\begin{aligned} & \frac{d}{dt}\int_M (|\nabla\bar{u}|^2 - \Delta\bar{u})dV_{\bar{g}} = \\ = \ & -\int_M |\nabla\nabla\bar{u}|^2 + \int_M |\nabla\bar{u}|^2 + \int_M (|\nabla\bar{u}|^2 - \Delta\bar{u})\Delta\bar{u}dV_{\bar{g}(t)} \\ = \ & -\int_M |\nabla\nabla\bar{u}|^2 dV_{h(t)}, \end{aligned} \tag{9}$$

where we have used the equation (8). On the other hand, we have that $\frac{d}{dt}\int_M(|\nabla\bar{u}|^2 - \Delta\bar{u})dV_t = \frac{d}{dt}\int_M(\bar{u}+a)$. Since $\frac{d}{dt}\bar{u}(t) = |\nabla\bar{u}|^2$, we have

$$\frac{d}{dt}\int_M \bar{u}dV_{\bar{g}(t)} = \int_M (|\nabla\bar{u}|^2 + \bar{u}\Delta\bar{u})dV_{\bar{g}(t)} = 0.$$

By claim 17, $\bar{a}(t) = a$ is independent of t and volume is fixed along the flow. These imply that $\frac{d}{dt}\int_M(\bar{u}+a)dV_{\bar{g}(t)} = 0$. From equation (9) we get

$$\int_M |\nabla\nabla\bar{u}|^2 dV_{\bar{g}} = 0,$$

i.e. $\bar{u}_{ij} = \bar{u}_{\bar{i}\bar{j}} = 0$.

□

Claim 18 tells us that $\bar{u}(t)$ comes from a holomorphic vector field on Y, i.e. $\bar{g}(t)$ is a Kähler-Ricci soliton for $t \in [0,\delta]$. Since our δ depends on the uniform bouund on the Ricci curvature and not on an initial time, we can apply the above proof of Theorem 2 to a sequence $t_i + \delta$ instead of a sequence t_i (with the same choice of δ as in $(*)$ above) to conclude that there exists a subsequence t_i such that $g(t_i+t) \to \bar{g}(t)$ and $u(t_i+t) \stackrel{C^{2,\alpha}}{\to} \bar{u}(t)$ uniformly on compact subsets of $Y\backslash\{p\} \times [0,2\delta]$ and such that $(g_{k\bar{j}})_t = g_{k\bar{j}} - R_{k\bar{j}} = \partial_k\partial_{\bar{j}}\bar{u}$ and $\bar{u}_{kj}(t) = 0$. Continuing this process and diagonalizing the sequence t_i we will get a subsequence t_i so that

$(M, g(t_i + t)) \to (Y, \bar{g}(t))$. The convergence is uniform on compact subsets of $Y \backslash \{p\} \times [0, \infty)$ and $\bar{g}(t)$ is a Kähler-Ricci soliton for all times t.

$\square$

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